# A Family of Maximal Algebras of Block Toeplitz Matrices

**Muhammad Ahsan Khan**

### Abstract

The maximal commutative subalgebras containing only Toeplitz matrices have been identified as generalized circulants. A similar simple description cannot be obtained for block Toeplitz matrices. We introduce and investigate certain families of maximal commutative algebras of block Toeplitz matrices.

## 1 Introduction

Toeplitz matrices are one of the most well-studied and understood classes of structured matrices, that arise naturally in several fields of mathematics, as well as applied areas as signal processing or time series analysis. The subject is many decades old; among the monographs dedicated to the subject are [8], [7], and [11].

The related area of block Toeplitz matrices is less studied, one of the reasons being the new difficulties that appear with respect to the scalar case. Besides its theoretical interest, the subject is also important in view of the applications to multivariate control theory. As references for block Toeplitz matrices one can use [1] and [6].

Multiplication properties of Toeplitz matrices have been discussed in [10, 9]. Since the product of two Toeplitz matrices is not necessarily a Toeplitz matrix,

it is interesting to investigate subsets of Toeplitz matrices that are closed under multiplication—that is, algebras of Toeplitz matrices. It is not obvious that such nontrivial algebras exist (besides scalar multiples of the identity); but it turns out that one can identify all such maximal algebras [10]; they are given by the so-called generalized circulants [5].

The investigation of the similar problem for block Toeplitz matrices has been proposed as an open question in [10], and it does not appear that much progress has been done since. The general problem of characterizing all maximal algebras of block Toeplitz matrices seems very hard. In the current paper we make a few steps in this direction, describing a certain family of such algebras.

The plan of the paper is the following. We start with a section of preliminaries. Then, as we will be interested in block Toeplitz matrices whose entries are themselves elements of maximal commutative subalgebras of matrices, we discuss in Section 3 several such examples. In Section 4 we introduce a certain family of maximal algebras that we will consider and prove some basic results. Finally, in the last three sections we discuss this family corresponding to the different maximal commutative subalgebras considered in Section 3.

## 2    Preliminaries

As usual, $\mathbb{C}$ will stand for the complex plane. We designate the algebra of all $d \times d$ matrices with entries from $\mathbb{C}$ by $\mathcal{M}_{d \times d}(\mathbb{C})$. An algebra is a vector space $\mathcal{A}$ over $\mathbb{C}$ which is closed to multiplication. A subset $\mathcal{B}$ of an algebra is called a subalgebra if it is itself an algebra; it is commutative if $ab = ba$ for every $a, b \in \mathcal{B}$. A subalgebra $\mathcal{B}$ of $\mathcal{A}$ is maximal commutative if for any commutative subalgebra $\mathcal{B}_1$ the inclusion $\mathcal{B} \subset \mathcal{B}_1$ implies that $\mathcal{B} = \mathcal{B}_1$.

If $\mathcal{B}$ is a subalgebra of $\mathcal{M}_{d \times d}(\mathbb{C})$, then $\mathcal{B}'$ denotes the commutant of $\mathcal{B}$ (the set of all $d \times d$ matrices commuting with every element of $\mathcal{B}$). It is not difficult to see that $\mathcal{B}'$ is also an algebra.

For diagonal matrices we will use the notation

$$diag\begin{pmatrix} a_1 & a_2 & \cdots & a_d \end{pmatrix} = \begin{pmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & a_d \end{pmatrix}$$

Classical *Toeplitz matrices* are square matrices whose entries are constant parallel to the main diagonal; that is, they are of the form

$$T = (t_{p-q})_{p,q=0}^{n-1} : t_j \in \mathbb{C}.$$

The linear space of all $n \times n$ Toeplitz matrices will be denoted by $\mathcal{T}_n$.

If we replace the scalar entries of Toeplitz matrices by $d \times d$ matrices, we obtain *block Toeplitz matrices*. Thus a block Toeplitz matrix is actually an $nd \times nd$ matrix, but which has been decomposed in $n^2$ blocks of dimension $d$, and these blocks are constant parallel to the main diagonal. We will use the notation $\mathbf{T} = (T_{p-q})_{p,q=0}^{n-1}$, for a block Toeplitz matrix, and we will denote by $\mathcal{T}_{n,d}$ the linear space of all $n \times n$ block Toeplitz matrices whose entries are $d \times d$ matrices; thus

$$\mathcal{T}_{n,d} = \left\{ (T_{p-q})_{p,q=0}^{n-1} : T_j \in \mathcal{M}_{d \times d}(\mathbb{C}) \right\}.$$

The scalar case is $\mathcal{T}_n = \mathcal{T}_{n,1}$.

It is well known that the product of two scalar Toeplitz matrices is not necessarily a Toeplitz matrix. The main multiplicative properties of $\mathcal{T}_n$ can be found in [10, Theorem 1.3]. The next two theorems summarize then in a convenient form.

**Theorem 2.1.** *The following statements are equivalent:*

*1. $\mathcal{A}$ is a maximal subalgebra of $\mathcal{T}_n$.*

*2. $\mathcal{A}$ is a maximal commutative subalgebra of $\mathcal{T}_n$.*

*3. There exist $a, b \in \mathbb{C}$, not both zero, such that*

$$\mathcal{A} = \left\{ T = \begin{pmatrix} t_0 & bt_1 & \ldots & bt_{n-1} \\ at_{n-1} & t_0 & \ldots & bt_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ at_1 & at_2 & \ldots & t_0 \end{pmatrix} : t_0, \ldots, t_{n-1} \in \mathbb{C} \right\}. \qquad (2.1)$$

It is immediate that the algebra $\mathcal{A}$ defined by (2.1) depends only on the quotient $a/b$. Shalom denotes, for any complex number $\alpha$, by $\Pi_\alpha$ the set of all Toeplitz matrices of the form (2.1), where $a = 1$, $b = \alpha$ and by $\Pi_\infty$ the set of all Toeplitz matrices of the form (2.1), where $a = 0$ and $b = 1$ (i.e. the set of all upper triangular Toeplitz matrices). These algebras are called *generalized circulants*.

**Theorem 2.2.** *If $T \in \mathcal{T}_n$ is invertible, then there exists $\alpha \in \mathbb{C} \cup \{\infty\}$ such that $T$ as well as $T^{-1}$ belong to $\Pi_\alpha$.*

As noted in [10], there is no satisfactory analog of the above results for block Toeplitz algebras, and a classification of maximal commutative subalgebras seems beyond reach. Our purpose in this paper is to explore a class of maximal commutative subalgebras of the subspace of block Toeplitz matrices.

## 3   Maximal commutative subalgebras of matrices

We have already noted that the class of block Toeplitz matrices is not closed with respect to multiplication. In fact, the next lemma can be proved after some slightly tedious computations.

**Lemma 3.1.** *Suppose $\mathbf{T} = (T_{p-q})_{p,q=0}^{n-1}$ and $\mathbf{U} = (U_{p-q})_{p,q=0}^{n-1}$ are two block Toeplitz matrices with entries in $\mathcal{M}_{d \times d}(\mathbb{C})$. The product $\mathbf{TU}$ is a block Toeplitz matrix if and only if*

$$T_p U_{q-n} = T_{p-n} U_q \quad \text{for all } p, q = 1, 2, \cdots n - 1. \tag{3.1}$$

The main purpose of the present article is to study certain subalgebras of $\mathcal{T}_{n,d}$. Their form is suggested by the generalized circulants that appear in Theorem 2.1; we will see however, first that some care must taken with their definition, and secondly that they do not exhaust all subalgebras of $\mathcal{T}_{n,d}$.

As a preparation, let us remember that the entries of scalar Toeplitz matrices are complex numbers. Therefore, as a first step in the study of subalgebras of block Toeplitz matrices, we will assume that their entries belong to a fixed maximal commutative subalgebra of $\mathcal{M}_{d \times d}(\mathbb{C})$, that we will denote by $\mathcal{B}$. There are many examples of maximal commutative subalgebras of $\mathcal{M}_{d \times d}(\mathbb{C})$,

and their classification is far from being achieved (see, for instance, [2, 3, 4]). Here is a non-exhaustive list to which we will refer in the sequel.

1. If we fix a basis in $\mathbb{C}^d$, then the algebra of diagonal matrices $\mathcal{D}$ is maximal commutative.

2. The generalized circulant algebras $\Pi_\alpha$ defined in section 2 are maximal commutative subalgebras. In particular, $\Pi_0$ is the algebra of lower triangular scalar Toeplitz matrices, while $\Pi_\infty$ is the algebra of upper triangular scalar Toeplitz matrices.

3. Fix positive integers $\sigma, \tau$, such that $\sigma + \tau = d$. Consider the family of matrices that have with respect to the decomposition $\mathbb{C}^d = \mathbb{C}^\sigma \oplus \mathbb{C}^\tau$ the form

$$\mathcal{O}_{\sigma,\tau} = \left\{ \begin{pmatrix} \lambda I_\sigma & X \\ 0 & \lambda I_\tau \end{pmatrix} \,\Big|\, \lambda \in \mathbb{C}, X \in \mathcal{M}_{\sigma \times \tau}(\mathbb{C}) \right\}$$

It is shown in [2] that, for $|\sigma - \tau| \leq 1$, $\mathcal{O}_{\sigma,\tau}$ is a maximal commutative algebra. We will call it a *Schur algebra*.

4. Suppose that $M$ is a nonderogatory matrix; that is, its minimal polynomial is equal to its characteristic polynomial. Then the algebra $\mathcal{P}(M)$ generated by $M$ is maximal commutative.

These examples show the large variety of maximal commutative subalgebras of matrices. We will use them as main test cases for our development.

Let us note, for further use, the following simple result. Recall that a subalgebra $\mathcal{B}$ is said to be inverse-closed if, whenever $B \in \mathcal{B}$ and $B$ is invertible, it follows that $B^{-1}$ also is in $\mathcal{B}$.

**Lemma 3.2.** *If $\mathcal{B}$ is a maximal commutative subalgebra of $\mathcal{M}_{d \times d}(\mathbb{C})$, then $\mathcal{B}$ is inverse closed.*

*Proof.* If $\mathcal{B}$ is a commutative subalgebra, then $\mathcal{B} \subset \mathcal{B}'$, and $\mathcal{B}$ is a maximal commutative subalgebra if and only if $\mathcal{B} = \mathcal{B}'$. Suppose then $A \in \mathcal{B}$ is invertible. Since $AB = BA$ implies $A^{-1}B = BA^{-1}$, it follows that $A^{-1} \in \mathcal{B}' = \mathcal{B}$. $\square$

## 4   A class of subalgebras

Fixing a maximal commutative subalgebra $\mathcal{B}$ in $\mathcal{M}_{d \times d}(\mathbb{C})$, we will denote by $\mathcal{T}_{n,d}(\mathcal{B})$ the linear space of block Toeplitz matrices with entries in $\mathcal{B}$. We are interested in identifying maximal subalgebras of $\mathcal{T}_{n,d}$ that are contained in $\mathcal{T}_{n,d}(\mathcal{B})$.

The class we will consider is inspired by the generalized circulants $\Pi_\alpha$ in Section 2. We start by noting that we can also define

$$\Pi_\alpha = \{T = (t_{p-q})_{p,q=0}^{n-1} : at_j = bt_{j-n} \text{ for } j = 1, \ldots, n-1\}.$$

Let us then fix $A, B \in \mathcal{B}'$, and define the family $\mathcal{F}_{A,B}^{\mathcal{B}}$ by

$$\mathcal{F}_{A,B}^{\mathcal{B}} = \left\{ \mathbf{T} = (T_{p-q})_{p,q=0}^{n-1} : T_j \in \mathcal{B}, AT_j = BT_{j-n}, j = 1, 2, \cdots n-1 \right\}. \quad (4.1)$$

We will use the following simple lemma.

**Lemma 4.1.** *Suppose $A, B \in \mathcal{M}_{d \times d}(\mathbb{C})$ satisfy the condition*

$$Ker A \cap Ker B = \{0\}. \quad (4.2)$$

*If $T$ is any $d \times d$ matrix such that $AT = BT = 0$ then $T = 0$.*

*Proof.* To prove $T = 0$ we will show that $Tx = 0$ for all $x \in \mathbb{C}^d$. The assumption that $ATx = 0$ and $BTx = 0$ for all $x \in \mathbb{C}^d$ implies that $Tx \in Ker A$ and $Tx \in Ker B$. It follows that $Tx \in Ker A \cap Ker B$. But since we have $Ker A \cap Ker B = \{0\}$, it follows that $Tx = 0$ and hence $T = 0$. $\square$

We obtain then the main result of this section.

**Theorem 4.2.** *Suppose $\mathcal{B}$ is a commutative subalgebra of $\mathcal{M}_{d \times d}(\mathbb{C})$, and $A, B \in \mathcal{B}'$. Then the family $\mathcal{F}_{A,B}^{\mathcal{B}}$ is a commutative linear subspace of $\mathcal{T}_{n,d}(\mathcal{B})$. If $A, B$ satisfy condition (4.2), then $\mathcal{F}_{A,B}^{\mathcal{B}}$ is an algebra.*

*Proof.* It is easy to see that $\mathcal{F}_{A,B}^{\mathcal{B}}$ is a commutative linear space. Let us then assume that condition (4.2) is satisfied. To show that $\mathcal{F}_{A,B}^{\mathcal{B}}$ is closed under matrix multiplication, suppose that $\mathbf{T}$ and $\mathbf{U}$ are two arbitrary elements of $\mathcal{F}_{A,B}^{\mathcal{B}}$; so

$$\begin{aligned} \mathbf{T} &= (T_{p-q})_{p,q=0}^{n-1}, \quad AT_j = BT_{j-n} \quad j = 1, 2, \cdots, n-1, \\ \mathbf{U} &= (U_{p-q})_{p,q=0}^{n-1}, \quad AU_j = BU_{j-n} \quad j = 1, 2, \cdots, n-1. \end{aligned} \quad (4.3)$$

The entry at the position $(i,j)$ of $\mathbf{TU}$ is

$$(\mathbf{TU})_{i,j} = \sum_{k=0}^{n-1} T_{i-k}U_{k-j}, \quad i,j = 0,1,\cdots n-1, \tag{4.4}$$

and therefore

$$(\mathbf{TU})_{i,j} - (\mathbf{TU})_{i+1,j+1} = T_{i-n+1}U_{n-1-j} - T_{i+1}U_{-1-j}.$$

Multiplying with $A$ and with $B$ respectively and using formulas (4.3), it follows that

$$A((\mathbf{TU})_{i,j} - (\mathbf{TU})_{i+1,j+1}) = 0,$$
$$B((\mathbf{TU})_{i,j} - (\mathbf{TU})_{i+1,j+1}) = 0.$$

We may then apply Lemma 4.1 to conclude that

$$(\mathbf{TU})_{i,j} - (\mathbf{TU})_{i+1,j+1} = 0,$$

and thus $\mathbf{V} := \mathbf{TU} \in \mathfrak{T}_{n,d}(\mathcal{B})$. Therefore $\mathbf{V} = (V_{i-j})_{i,j=0}^{n-1}$.

To prove that $\mathbf{V} \in \mathcal{F}_{A,B}^{\mathcal{B}}$, consider $j = 0$ in (4.4). Then

$$AV_i = A(\mathbf{TU})_{i,0} = A(\sum_{k=0}^{n-1} T_{i-k}U_k) = \sum_{k=0}^{n-1} T_{i-k}AU_k$$

$$= \sum_{k=0}^{n-1} T_{i-k}BU_{k-n} = B(\sum_{k=0}^{n-1} T_{i-k}U_{k-n}) = B(\mathbf{TU})_{i,n} = BV_{i-n}.$$

Therefore $\mathbf{V} \in \mathcal{F}_{A,B}^{\mathcal{B}}$. This proves, along with the earlier fact that $\mathcal{F}_{A,B}^{\mathcal{B}}$ is a linear subspace, that $\mathcal{F}_{A,B}^{\mathcal{B}}$ is an algebra in $\mathfrak{T}_{n,d}$.     $\square$

**Remark 4.3.** Condition (4.1) has been shown to imply that $\mathcal{F}_{A,B}^{\mathcal{B}}$ is an algebra. However, the converse is not true; an example will appear in Section 6 below.

The next result concerns the maximality of $\mathcal{F}_{A,B}^{\mathcal{B}}$ as a commutative algebra in $\mathfrak{T}_{n,d}$.

**Theorem 4.4.** *Let $\mathcal{B}$ be a commutative subalgebra of $\mathcal{M}_{d\times d}(\mathbb{C})$. The following assertions are equivalent:*

1. $\mathcal{B}$ *is a maximal commutative subalgebra of* $\mathcal{M}_{d \times d}(\mathbb{C})$.

2. *The family* $\mathcal{F}_{A,B}^{\mathcal{B}}$ *is a maximal commutative algebra in* $\mathcal{T}_{n,d}$.

*Proof.* Suppose that $\mathcal{B}$ is a maximal commutative subalgebra of $\mathcal{M}_{d \times d}(\mathbb{C})$. Suppose that $\mathcal{F}$ is a commutative algebra in $\mathcal{T}_{n,d}$ such that $\mathcal{F}_{A,B}^{\mathcal{B}} \subset \mathcal{F}$. Let $\mathbf{T} = (T_{p-q})_{p,q=0}^{n-1}$ be any arbitrary element of $\mathcal{F}$. For an arbitrary $U \in \mathcal{B}$ define $\mathbf{U} = diag \left( U \quad U \quad \cdots U \right)$. Then $\mathbf{U} \in \mathcal{F}_{A,B}^{\mathcal{B}}$. Since $\mathcal{F}_{A,B}^{\mathcal{B}} \subset \mathcal{F}$ and $\mathcal{F}$ is a commutative algebra in $\mathcal{T}_{n,d}$, we have $\mathbf{TU} = \mathbf{UT}$. Comparing corresponding entries of $\mathbf{TU}$ and $\mathbf{UT}$, we obtain $T_p U = U T_p$ for all $p$, $|p| \leq n-1$. Since $U \in \mathcal{B}$ and $\mathcal{B}$ is a maximal commutative subalgebra of $\mathcal{M}_{d \times d}(\mathbb{C})$, it follows that $T_p \in \mathcal{B}$ for all $p$.

Now let $\mathbf{T} \in \mathcal{F}$ be any arbitrary element of the form $\mathbf{T} = (T_{p-q})_{p,q=0}^{n-1}$ and consider the fixed element $\mathbf{J} \in \mathcal{F}_{A,B}^{\mathcal{B}}$ defined by

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 0 & \ldots & A \\ B & 0 & 0 & \ldots & 0 \\ 0 & B & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & B & 0 \end{pmatrix}$$

Using formula (4.4) for $\mathbf{JT}$, one obtains

$$(\mathbf{JT})_{i,j} = \begin{cases} AT_{n-1-j} & \text{for } i = 0, \quad j = 0, \ldots, n-1, \\ BT_{i-1-j} & \text{for } i \neq 0, \quad j = 0, \ldots, n-1. \end{cases} \tag{4.5}$$

Since $\mathcal{F}$ is contained in $\mathcal{T}_{n,d}$, the entries along the main diagonals of $\mathbf{JT}$ should be equal. In particular,

$$(\mathbf{JT})_{0,j} = (\mathbf{JT})_{1,j+1}$$

for $j = 1, \ldots, n-2$. By (4.5) this means $AT_{n-1-j} = BT_{-j-1}$, or, by denoting $i = n-1-j$,

$$AT_i = BT_{i-n}$$

for $i = 1, \ldots, n-1$. Therefore $\mathbf{T} \in \mathcal{F}_{A,B}^{\mathcal{B}}$. Consequently $\mathcal{F} \subset \mathcal{F}_{A,B}^{\mathcal{B}}$. whence $\mathcal{F}_{A,B}^{\mathcal{B}}$ is a maximal commutative algebra in $\mathcal{T}_{n,d}$.

Conversely, suppose that $\mathcal{F}_{A,B}^{\mathcal{B}}$ is a maximal commutative algebra in $\mathcal{T}_{n,d}$. If $\mathcal{B}_1$ is a subalgebra of $\mathcal{M}_{d \times d}(\mathbb{C})$ such that $\mathcal{B} \subset \mathcal{B}_1$, then $\mathcal{F}_{A,B}^{\mathcal{B}} \subset \mathcal{F}_{A,B}^{\mathcal{B}_1}$. The

maximality of $\mathcal{F}_{A,B}^{\mathcal{B}}$ implies that $\mathcal{F}_{A,B}^{\mathcal{B}} = \mathcal{F}_{A,B}^{\mathcal{B}_1}$, whence it follows easily that $\mathcal{B} = \mathcal{B}_1$. Therefore $\mathcal{B}$ is a maximal commutative subalgebra of $\mathcal{M}_{d \times d}(\mathbb{C})$.  $\square$

As a consequence of Theorem 4.4, we will consider in the sequel only maximal commutative algebras $\mathcal{B}$, whence $\mathcal{B} = \mathcal{B}'$. We will therefore take in the definition of the algebras $\mathcal{F}_{A,B}^{\mathcal{B}}$ matrices $A, B \in \mathcal{B}$.

These algebras include an important general class of maximal block Toeplitz algebras, as shown by the next result.

**Theorem 4.5.** *Let $\mathcal{B}$ a maximal commutative subalgebra of $\mathcal{M}_{d \times d}(\mathbb{C})$ and $\mathcal{A}$ an algebra contained in $\mathfrak{T}_{n,d}(\mathcal{B})$. Suppose $\mathcal{A}$ contains an element $\mathbf{T}$ such that $T_r$ is invertible for some $r \neq 0$. Then $\mathcal{A} \subset \mathcal{F}_{A,B}^{\mathcal{B}}$ for some $A$ and $B$.*

*Proof.* Let $\mathbf{T} = (T_{p-q})_{p,q=0}^{n-1}$ and $\mathbf{U} = (U_{p-q})_{p,q=0}^{n-1}$ be two elements of $\mathcal{A}$. Since $\mathcal{A}$ is an algebra contained in $\mathfrak{T}_{n,d}$, the product $\mathbf{TU} \in \mathcal{A}$ if and only if

$$T_p U_{q-n} = T_{p-n} U_q \quad (q = 1, 2, \cdots n-1). \tag{4.6}$$

Suppose $r > 0$. Since we have assumed that $T_r$ is invertible, it follows from (4.6) that

$$U_{q-n} = T_r^{-1} T_{r-n} U_q \quad (q = 1, 2, \cdots n-1).$$

The subalgebra $\mathcal{B}$ is inverse closed by Lemma 3.2, and so $T_r^{-1} T_{r-n} \in \mathcal{B}$. It follows then that, if we take $A = I$ and $B = T_r^{-1} T_{r-n}$, then $\mathcal{A} \subset \mathcal{F}_{A,B}^{\mathcal{B}}$. A simlar argument works for $r < 0$, finishing the proof.  $\square$

**Corollary 4.6.** *Let $\mathcal{B}$ a maximal commutative subalgebra of $\mathcal{M}_{d \times d}(\mathbb{C})$ and $\mathcal{A}$ a maximal subalgebra of $\mathfrak{T}_{n,d}(\mathcal{B})$. Suppose $\mathcal{A}$ contains an element $\mathbf{T}$ such that $T_p$ is invertible for some $p \neq 0$. Then $\mathcal{A} = \mathcal{F}_{A,B}^{\mathcal{B}}$ for some $A$ and $B$.*

In the next sections we will investigate the algebras $\mathcal{F}_{A,B}^{\mathcal{B}}$ corresponding to several cases of maximal subalgebras $\mathcal{B}$.

## 5   Diagonal entries

To discuss the cases of entries belonging to the algebra of diagonal matrices with respect to a given basis, we use the next lemma, which describes the structure of such block Toeplitz matrices.

**Lemma 5.1.** *Suppose* $\mathbf{T} = (T_{p-q})_{p,q=0}^{n-1}$ *is a block Toeplitz matrix, such that each $T_j$ is diagonal. Then there is a change of basis that brings $\mathbf{T}$ into the following form*

$$\mathbf{T}' = diag \begin{pmatrix} T_1' & T_2' & \cdots T_d' \end{pmatrix},$$

*where for every $k = 1, 2, \cdots d$, $T_k'$ is a scalar Toeplitz matrix of order $n$.*

*Proof.* Suppose that the original basis is

$$\mathcal{E} = \{e_1^0, e_2^0, \cdots e_d^0, e_1^1, e_2^1, \cdots e_d^1, \cdots e_1^{n-1}, e_2^{n-1}, \cdots e_d^{n-1}\},$$

and $T_j = diag \begin{pmatrix} t_{j1} & t_{j2} & \cdots t_{jd} \end{pmatrix}$ for $j = 0, \pm 1, \cdots \pm (n-1)$. The new basis is then

$$\mathcal{E}' = \{e_1^0, e_1^1, \cdots e_1^{n-1}, e_2^0, e_2^1, \cdots e_2^{n-1}, \cdots e_d^0, e_d^1, \cdots e_d^{n-1}\}$$

(it is obtained by "reshuffling"). With respect to $\mathcal{E}'$, $\mathbf{T}$ has the required form $\mathbf{T}' = diag \begin{pmatrix} T_1' & T_2' & \cdots T_d' \end{pmatrix}$, where, for each $k = 1, 2, \cdots d$, $T_k' = (t_{r-s,k})_{r,s=0}^n$, i.e. $T_k'$ is a Toeplitz matrix of order $n$. $\square$

It is clear from the proof that $\mathcal{E}'$ depends only on the basis $\mathcal{E}$, not on the particular matrix $\mathbf{T}$.

**Proposition 5.2.** *Suppose that $\mathcal{A}$ is any commutative algebra of block Toeplitz matrices contained in $\mathfrak{T}_{n,d}(\mathcal{D})$. Then $\mathcal{A} \subset \mathcal{F}_{A,B}^{\mathcal{D}}$ for some $A, B \in \mathcal{D}$.*

*Proof.* Applying Lemma 5.1, we obtain a basis $\mathcal{E}'$ with respect to which any element $\mathbf{T} \in \mathcal{A}$ has the form $\mathbf{T}' = diag \begin{pmatrix} T_1' & T_2' & \cdots T_d' \end{pmatrix}$, where for each $k = 1, 2, \cdots d$ we have $T_k' = (t_{r-s,k})_{r,s=0}^d$; that is, $T_k'$ is a scalar Toeplitz matrix. Since $\mathcal{A}$ is closed with respect to multiplication, each of the component blocks has to be closed with respect to multiplication. It follows that for every $k = 1, 2, \cdots d$, $T_k' \in \Pi_{\alpha_k}$ for some $\alpha_k \in \mathbb{C} \cup \{\infty\}$.

We may then define $B = diag \begin{pmatrix} b_1 & b_2 & \cdots & b_d \end{pmatrix}, A = diag \begin{pmatrix} a_1 & a_2 & \cdots & a_d \end{pmatrix}$ as follows:

— if $\alpha_k \neq \infty$, then $a_k = \alpha_k$ and $b_k = 1$;

— if $\alpha_k = \infty$, then $a_k = 1$ and $b_k = 0$.

Then $Ker A \cap Ker B = \{0\}$, and it is easily checked that $\mathcal{A} \subset \mathcal{F}_{A,B}^{\mathcal{D}}$. $\square$

**Corollary 5.3.** *If $\mathcal{A}$ is a maximal subalgebra of $\mathfrak{T}_{n,d}(\mathcal{D})$, then $\mathcal{A} = \mathcal{F}_{A,B}^{\mathcal{D}}$ for some $A, B \in \mathcal{D}$.*

## 6    Entries in the Schur algebras

We start with the following simple lemma.

**Lemma 6.1.** *An element $T \in \mathcal{O}_{\sigma,\tau}$ is invertible if and only if $\lambda \neq 0$.*

*Proof.* Let $T \in \mathcal{O}_{\sigma,\tau}$ be an element of the form

$$T = \begin{pmatrix} \lambda I_\sigma & X \\ 0 & \lambda I_\tau \end{pmatrix}.$$

Then $T$ is invertible if and only if $\det(T) = \lambda^d \neq 0$ if and only if $\lambda \neq 0$.    □

Suppose now that $\mathcal{A}$ is a maximal commutative subalgebra that has its entries in the Schur algebra $\mathcal{O}_{\sigma,\tau}$. If at least one element of $\mathcal{A}$ has an invertible off-diagonal entry, then we can apply Theorem 4.5 and obtain that $\mathcal{A} = \mathcal{F}_{A,B}^{\mathcal{O}_{\sigma,\tau}}$ for some $A, B \in \mathcal{O}_{\sigma,\tau}$. The next theorem shows that there is only one other maximal commutative subalgebra contained in $\mathcal{T}_{n,d}(\mathcal{O}_{\sigma,\tau})$.

**Theorem 6.2.** *The set $\mathcal{S}$ of block Toeplitz matrices that have entries in $\mathcal{O}_{\sigma,\tau}$, and all off diagonal entries are noninvertible forms a maximal commutative algebra. It is of type $\mathcal{F}_{A,B}^{\mathcal{B}}$ for some $A, B$ that do not satisfy condition* (4.2).

*Proof.* First we will show that $\mathcal{S}$ is an algebra. Clearly $\mathcal{S}$ is a linear subspace. we have to show only that it is closed under block matrix multiplication. For this let $\mathbf{T} = (T_{p-q})_{p,q=0}^{n-1}$ and $\mathbf{U} = (U_{p-q})_{p,q=0}^{n-1}$ be any two arbitrary elements of $\mathcal{S}$ then $\mathbf{TU} \in \mathcal{S}$ if and only if

$$T_p U_{q-n} = T_{p-n} U_q \quad (p, q = 1, 2, \cdots n - 1).$$

Since each $T_i$ and $U_i$ is noninvertible, lemma 6.1 implies that for every $i \neq 0$ $T_i$ and $U_i$ are strictly upper triangular $2 \times 2$ block matrices. Therefore

$$T_p U_{q-n} = T_{p-n} U_q = 0 \quad (p, q = 1, 2, \cdots n - 1), \tag{6.1}$$

whence it follows that $\mathbf{TU} \in \mathcal{S}$.

We will show that $\mathcal{S}$ is a maximal commutative subalgebra contained in $\mathcal{T}_{n,d}(\mathcal{O}_{\sigma,\tau})$. Suppose $\mathbf{U} = (U_{p-q})_{p,q=0}^{n-1}$ commutes with $\mathcal{S}$ and denote

$$U_k = \begin{pmatrix} \lambda_k I_\sigma & X_k \\ 0 & \lambda_k I_\tau \end{pmatrix}, \quad \lambda_k \in \mathbb{C} \quad , X_k \in \mathcal{M}_{\sigma \times \tau}(\mathbb{C}),$$

Suppose, for instance, that $k \geq 1$. Take then $\mathbf{T} = (T_{p-q})_{p,q=0}^{n-1} \in \mathcal{S}$ defined by

$$T_p = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad T_{p-n} = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}, \quad p = 1, 2, \cdots n-1,$$

with $Y$ some nonzero matrix in $\mathcal{O}_{\sigma,\tau}$. Then

$$T_p U_{k-n} = 0, \quad T_{p-n} U_k = \begin{pmatrix} 0 & \lambda_k \\ 0 & 0 \end{pmatrix}, \quad p = 1, 2, \cdots n-1.$$

From (3.1) it follows then that $\lambda_k = 0$. A similar argument works for $k < 0$. Therefore $\mathbf{U} \in \mathcal{S}$, whence $\mathcal{S}$ is maximal commutative.

To obtain $\mathcal{S} = \mathcal{F}_{A,B}^{\mathcal{O}_{\sigma,\tau}}$, take two linearly independent matrices $X, Y \in \mathcal{O}_{\sigma,\tau}$, and define

$$A = \begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix}.$$

If $\mathbf{T} = (T_{p-q})_{p,q=0}^{n-1} \in \mathcal{S}$, then $AT_j = BT_{j-n} = 0$ for all $j = 1, \ldots, n$. So $\mathcal{S} \subset \mathcal{F}_{A,B}^{\mathcal{O}_{\sigma,\tau}}$.

To prove the reverse, take $\mathbf{T} = (T_{p-q})_{p,q=0}^{n-1} \in \mathcal{F}_{A,B}^{\mathcal{O}_{\sigma,\tau}}$. If

$$T_i = \begin{pmatrix} \lambda_i & X_i \\ 0 & \lambda_i \end{pmatrix},$$

then

$$AT_i = \begin{pmatrix} 0 & \lambda_i X \\ 0 & 0 \end{pmatrix}, \quad BT_{i-n} = \begin{pmatrix} 0 & \lambda_{i-n} Y \\ 0 & 0 \end{pmatrix}.$$

Since $X, Y$ are linearly independent, condition (4.1) implies $\lambda_i = \lambda_{i-n} = 0$ for $i = 1, \ldots, n-1$. Therefore $\mathbf{T}$ is not invertible, so it belongs to $\mathcal{O}_{\sigma,\tau}$.    $\square$

It is interesting to note that in this case $\mathcal{F}_{A,B}^{\mathcal{B}}$ is an algebra, although $A, B$ do not satisfy condition (4.2). Note that we cannot write $\mathcal{S} = \mathcal{F}_{A,B}^{\mathcal{B}}$ with some $A, B$ satisfying condition (4.2). Indeed, for a noninvertible matrix $A \in \mathcal{O}_{\sigma,\tau}$ we have $\mathbb{C}^\sigma \oplus 0 \in \ker A$. Therefore, to satisfy condition (4.2), at least one of $A$ and $B$ has to be invertible. Suppose then that

$$A = \begin{pmatrix} \lambda & X \\ 0 & \lambda \end{pmatrix}, \quad B = \begin{pmatrix} \mu & Y \\ 0 & \mu \end{pmatrix}.$$

For a noninvertible $\mathbf{T} = (T_{p-q})_{p,q=0}^{n-1}$, with

$$T_i = \begin{pmatrix} 0 & X_i \\ 0 & 0 \end{pmatrix},$$

condition (4.1) becomes

$$\begin{pmatrix} 0 & \lambda X_i \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mu X_{i-n} \\ 0 & 0 \end{pmatrix},$$

or $\lambda X_i = \mu X_{i-n}$ If at least one of $\lambda, \mu$ is nonzero, this equality is not satisfied by every $\mathbf{T} \in \mathcal{S}$.

**Corollary 6.3.** *If $\mathcal{A}$ is a maximal subalgebra of $\mathcal{T}_{n,d}(\mathcal{O}_{\sigma,\tau})$, then $\mathcal{A} = \mathcal{F}_{A,B}^{\mathcal{O}_{\sigma,\tau}}$ for some $A, B \in \mathcal{O}_{\sigma,\tau}$.*

## 7    Entries in a singly generated algebra

Suppose $M$ is a nonderogatory matrix, with minimal polynomial $p_M$ of degree $\delta$, and $\mathcal{B}$ is the algebra generated by $M$, that is

$$\mathcal{B} = \mathcal{P}(M) := \{p(M) : p \text{ polynomial } \} = \{p(M) : p \text{ polynomial }, \deg p < \delta\}.$$

The next lemma is well known.

**Lemma 7.1.** *An element $p(M) \in \mathcal{P}(M)$ is invertible if and only if $p$ and $p_M$ are relatively prime.*

As a consequence of Theorem 4.5, we have then the next result.

**Corollary 7.2.** *Suppose $\mathcal{A} \subset \mathcal{T}$ is an algebra with entries in $\mathcal{P}(M)$. Suppose $\mathcal{A}$ contains an element $\mathbf{T}$ such that for some $j \neq 0$, $T_j = p(M)$ with $p$ and $p_M$ relatively prime. Then $\mathcal{A} \subset \mathcal{F}_{A,B}^{\mathcal{P}(M)}$ for some $A, B \in \mathcal{P}(M)$.*

However, not all maximal subalgebras of $\mathcal{T}_{n,d}$ are of type $\mathcal{F}_{A,B}^{\mathcal{P}(M)}$. To see this, we will discuss in the rest of this section the particular case of a nilpotent matrix of order 2; that, we assume that $M \neq 0$ and $M^2 = 0$.

**Theorem 7.3.** *Suppose $M$ is a nonderogatory matrix such that $M^2 = 0$. The set $\mathcal{A}$ of all block Toeplitz matrices that have entries in $\mathcal{P}(M)$, and all off diagonal entries are noninvertible, form a maximal commutative algebra that is not of the type $\mathcal{F}_{A,B}^{\mathcal{P}(M)}$.*

*Proof.* It is easy to see that $\mathcal{A}$ is an algebra: if $\mathbf{T}, \mathbf{S} \in \mathcal{A}$, then any nondiagonal entry of $\mathbf{TS}$ is a sum of products, and each of the products contains at least one term that has $M$ as a factor, and is thus noninvertible.

To show that $\mathcal{A}$ is maximal commutative, suppose that $\mathbf{TU} = \mathbf{UT}$ for all $\mathbf{T} \in \mathcal{A}$. Take $\mathbf{T}$ with

$$T_p = 0, \quad T_{p-n} = M \text{ for } p = 1, \ldots, n. \tag{7.1}$$

Formula (3.1) tells us that $0 = MU_q$ for all $q = 1, \ldots, n$, whence it follows that $U_q$ cannot be invertible. Similarly, by taking $\mathbf{T}$ with

$$T_p = M, \quad T_{p-n} = 0 \text{ for } p = 1, \ldots, n, \tag{7.2}$$

we obtain $U_{q-n}$ is noninvertible for all $q = 1, \ldots, n$. Therefore $\mathbf{U}$ has all its off-diagonal entries noninvertible, and therefore it belongs to $\mathcal{A}$.

Suppose now that $\mathcal{A} = \mathcal{F}_{A,B}^{\mathcal{P}(M)}$ for some $A, B \in \mathcal{A}$. In particular, using again $\mathbf{T}$ from (7.1) it follows from (4.1) that $BM = 0$. Similarly, using (7.2) we obtain $AM = 0$. So $A$ and $B$ are noninvertible elements in $\mathcal{P}(M)$, say $A = aM$, $B = bM$.

If $a = b = 0$, then $\mathcal{F}_{A,B}^{\mathcal{P}(M)} = \mathcal{T}_{n,d}(\mathcal{P}(M)) \neq \mathcal{A}$. Otherwise, suppose at least one of $a, b$ is nonzero. If we define $\mathbf{T}$ by

$$T_j = bI_d, \quad T_{n-j} = aI_d,$$

and $T_i = 0$ for other values of $T$, Then $\mathbf{T}$ satisfies (4.1), and therefore $\mathbf{T} \in \mathcal{F}_{A,B}^{\mathcal{P}(M)}$. But obviously $\mathbf{T} \notin \mathcal{A}$, so $\mathcal{A} \neq \mathcal{F}_{A,B}^{\mathcal{P}(M)}$. $\qquad\square$

## Acknowledgements

## References

[1] A. Bottcher and B. Silbermann, Analysis of Toeplitz operators, Springer-Verlag, 1990.

[2] W.C. Brown, F.W. Call, Maximal commutative subalgebras of $n \times n$ matrices, Communications in Algebra 22 (1994), 4439-4460.

[3] W.C. Brown, Two constructions of maximal commutative subalgebras of $n \times n$ matrices, Communications in Algebra 22 (1994), 4051-4066.

[4] W.C. Brown, Constructing maximal commutative subalgebras of matrix rings in small dimensions, Communications in Algebra 25 (1997), 3923-3946.

[5] P.J. Davis, Circulant Matrices, Wiley, New York, 1979.

[6] I. Gohberg, S. Goldberg and M.A. Kaashoek, Classes of Linear Operators, Vol. 2, Birkhauser Verlag Basel, 1993.

[7] U. Grenander and G. Szego, Toeplitz Forms and Their Applications. University of Calif. Press, Berkeley and Los Angeles, 1958.

[8] I. S. Iohvidov, Hankel and Toeplitz Matrices and Forms, Birkhuser Boston, Cambridge, MA, 1982.

[9] C. Gu, L. Patton, Commutation relation for Toeplitz and Hankel matrices, SIAM J. Matrix Anal. Appl. 24 (2003), 728–746.

[10] T. Shalom, On Algebras of Toeplitz matrices, Linear Algebra and its applications 96 (1987), 211-226.

[11] H. Widom, Toeplitz Matrices, in Studies in Real and Complex Analysis, (J. I.I. Hirschmann, ed.), MAA Studies in Mathematics, Prentice-Hall, Englewood Cliffs, NJ, 1965.

Muhammad Ahsan Khan,
Abdus Salam School of Mathematical Sciences,
GC University,
Lahore, Pakistan.
Email: ahsan.khan@sms.edu.pk,smsahsankhan@gmail.com